Analyzing (another) rational function: \( f(x) = \frac{x^3 - 3x + 2}{2x^3 - 4x^2 - 2x + 4} \)

(1a) Factoring denominator:

\[
2x^3 - 4x^2 - 2x + 4 = 2(x^3 - 2x^2 - x + 2)
\]

\[
= 2(x^2(x - 2) - (x - 2))
\]

\[
= 2(x^2 - 1)(x - 2) = 2(x - 1)(x + 1)(x - 2)
\]

(1b) Factoring numerator: *not so easy.* On the other hand, by inspection \( x = 1 \) is a zero of \( p(x) = x^3 - 3x + 2 \), which means that \( (x - 1) \) is a factor of \( p(x) \), so we can write

\[
x^3 - 3x + 2 = (x - 1)(x^2 + bx + c) = x^3 + (b - 1)x^2 + (c - b)x - c
\]

\[
\implies c = -2 \quad \text{and} \quad b = 1
\]

I.e.,

\[
x^3 - 3x + 2 = (x - 1)(x^2 + x - 2) = (x - 1)(x - 1)(x + 2).
\]
Factored form of $f(x)$:

$$f(x) = \frac{x^3 - 3x + 2}{2x^3 - 4x^2 - 2x + 4} = \frac{(x - 1)(x - 1)(x + 2)}{2(x - 1)(x + 1)(x - 2)}.$$  

(2) Domain of $f(x)$: \{x\mid x \neq 1, -1, -2\}.

**Observation:** The factor $(x - 1)$ appears both in the numerator and the denominator...

(3) Reduced form of $f(x)$:

$$f(x) = \frac{(x - 1)(x - 1)(x + 2)}{2(x - 1)(x + 1)(x - 2)} = \frac{(x - 1)(x + 2)}{2(x + 1)(x - 2)} = f_r(x)$$

$$\Rightarrow f(x) = f_r(x) \text{ at every point } x \neq 1. \Leftarrow$$

The graph of $f(x)$ has a ‘hole’ at $x = 1$: 

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**Zeros.** The zeros of $f_r(x)$ are $x = 1$ and $x = -2$, but $f(x)$ is not defined at $x = 1$, so technically, $f(x)$ has only one zero, at $x = -2$. (We do consider $x = 1$ as a point where $f(x)$ may change sign, however).

**Vertical asymptotes.** These *may* occur at points where the function is not defined. There are three points to consider.

(*) At $x = -1$, the numerator of $f(x)$ is $4 \neq 0$ while the denominator is $0$, so the line $x = -1$ is a vertical asymptote.

(*) At $x = 1$, the reduced from $f_r(x)$ is defined at $x = 1$ ($f_r(1) = 0$), so there is no vertical asymptote there — we already know that there is a hole in the graph at this point.

(*) At $x = 2$, the numerator of $f(x)$ is $2 \neq 0$ and the denominator is $0$, so the line $x = 2$ is a vertical asymptote.
**Horizontal asymptote.** The numerator and denominator of $f_r$ have the same degree, 2, so there is a horizontal asymptote. If $|x|$ is large, then

$$f_r(x) = \frac{x^2 + x - 2}{2x^2 - 2x - 4} \approx \frac{x^2}{2x^2} = \frac{1}{2},$$

so the horizontal asymptote is the line $y = \frac{1}{2}$.

Asymptotes, zeros, $y$-intercept ($= 1/2$) and the hole at $x = 1$. 
Signs.

The sign of $f(x)$ can only change at: $-2$ (a zero); $-1$ (vertical asymptote); $1$ (hole in the graph); $2$ (vertical asymptote).

$\Rightarrow$ Sample $f_r(x)$ in the intervals...

(*) $(-\infty, -2)$: $f_r(-3) = 4/20 > 0$

(*) $(-2, -1)$: $f_r(-1.5) = (-1.75)/2.5 < 0$

(*) $(-1, 1)$: $f(0) = 1/2 > 0$

(*) $(1, 2)$: $f_r(1.5) = 1.75/(-2.5) < 0$

(*) $(2, \infty)$: $f_r(100) \approx 1/2 > 0$. 
Signs in each interval

-3 -2 -1 0 1 2 3 4 5 6 7 8 9
+ + +
- -

Graph showing the signs in each interval.
Putting it all together....