Section 3.2, Problem 52:

If \( p/q \) is a rational zero of the polynomial

\[
f(x) = 2x^4 - x^3 - 5x^2 + 2x + 2,
\]

then \( p \) must be a divisor of the constant coefficient 2, and \( q \) must be a divisor of the leading coefficient, which is also 2. I.e., \( p = \pm 1, \pm 2 \) and \( q = \pm 1, \pm 2 \), which means that the possible rational zeros are \( p/q = \pm 1, \pm 2 \) and \( \pm 1/2 \).

Now we test the candidates:

\[
f(1) = 0 \checkmark, \quad f(-1) = -2, \quad f(2) = 10, \quad f(-2) = 18, \quad f(1/2) = \frac{7}{4} \quad \text{and} \quad f(-1/2) = 0 \checkmark.
\]

This means that \( x = 1 \) and \( x = -1/2 \) are the rational zeros of \( f(x) \).

Next we use this information to factor \( f(x) \). We know that \((x - 1)\) and \((x + \frac{1}{2})\) are both factors of \( f(x) \), so \( f(x) = (x - 1)\left(x + \frac{1}{2}\right)g(x)\), and since \( f \) has degree 4, it follows that \( g \) must have degree 2, i.e., \( g(x) = ax^2 + bx + c \). This means that

\[
2x^4 - x^3 - 5x^2 + 2x + 2 = (x - 1)\left(x + \frac{1}{2}\right)(ax^2 + bx + c)
\]

\[
= (x^2 - \frac{1}{2}x - \frac{1}{2})(ax^2 + bx + c)
\]

\[
= ax^4 + \left(b - \frac{1}{2}a\right)x^3 + \left(c - \frac{1}{2}a - \frac{1}{2}b\right)x^2 - \frac{1}{2}(b + c)x - \frac{1}{2}c.
\]

Comparing coefficients, of the two degree four polynomials, we have

\[
a = 2, \quad b - \frac{1}{2}a = -1, \quad c - \frac{1}{2}(a + b) = -5, \quad -\frac{1}{2}(b + c) = 2 \quad \text{and} \quad -\frac{1}{2}c = 2.
\]

This means that \( a = 2 \) and \( c = -4 \), from which it follows that \( b = 0 \), so

\[
g(x) = (2x^2 - 4) = 2(x^2 - 2) = 2(x - \sqrt{2})(x + \sqrt{2})
\]

and \( f(x) \) factors as

\[
f(x) = 2(x - 1)\left(x + \frac{1}{2}\right)\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right).
\]

Section 3.4, Problem 54:

The degree of the numerator \( p(x) = x^4 - 16 \) is 2 more than the degree of the denominator \( q(x) = x^2 - 2x \), so the rational function

\[
F(x) = \frac{x^4 - 16}{x^2 - 2x}
\]

does not have an oblique or horizontal asymptote.

Next, factoring the numerator and the denominator, we have

\[
x^4 - 16 = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4) \quad \text{and} \quad x^2 - 2x = x(x - 2)
\]
so the reduced form of $F(x)$ is

$$F_r(x) = \frac{(x-2)(x+2)(x^2+4)}{x(x-2)} = \frac{(x+2)(x^2+4)}{x},$$

which means that the graph $y = F(x)$ has one vertical asymptote, $x = 0$ (and a hole at $(2, 16)$).

**Conclusion:** No horizontal or oblique asymptotes and one vertical asymptote, $x = 0$.

![Figure 1: The graph of $F(x) = \frac{x^4-16}{x^2-2x}$](image)

**Section 3.5, Problem 8:** Analyzing the graph of $R(x) = \frac{x}{(x-1)(x+2)}$.

**Step 1.** $R(x)$ is already factored above. The domain of $R(x)$ is $\{x| x \neq 1, -2\}$.

**Step 2.** $R(x)$ is already in lowest terms (i.e., the numerator and denominator don’t have any common factors).

**Step 3.** $R(x)$ has one zero, at $x = 0$, i.e., $R(0) = 0$. This is both the (only) $x$-intercept and the $y$-intercept. Since $x$ appears to an odd power in, the graph of $R(x)$ crosses the $x$-axis at this point.

**Step 4.** $R(x)$ has the lines $x = 1$ and $x = -2$ as vertical asymptotes.

(i) To the left of $x = -2$, $x < 0$, $x - 1 < 0$ and $x + 2 < 0$, so $R(x) < 0$ to the left of $x = -2$, and therefore $R(x) \to -\infty$ on the left side of the asymptote $x = -2$.

(ii) To the (immediate) right of $x = -2$, $x < 0$, $x - 1 < 0$ and $x + 2 > 0$, so $R(x) > 0$ to the right of $x = -2$, and therefore $R(x) \to +\infty$ on the right side of the asymptote $x = -2$.

(iii) To the (immediate) left of $x = 1$, $x > 0$, $x - 1 < 0$ and $x + 2 > 0$, so $R(x) < 0$ to the left of $x = 1$, and therefore $R(x) \to -\infty$ on the left side of the asymptote $x = 1$. 

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(iv) To the (immediate) right of \( x = 1 \), \( x > 0 \), \( x - 1 > 0 \) and \( x + 2 > 0 \), so \( R(x) > 0 \) to the right of \( x = 1 \), and therefore \( R(x) \to +\infty \) on the right side of the asymptote \( x = 1 \).

Step 5. The degree of the numerator is less than the degree of the denominator, so \( y = 0 \) (the \( x \)-axis) is a horizontal asymptote to the graph of \( R(x) \). As we already saw in Step 3., the graph intersects the \( x \)-axis at the origin and nowhere else.

Step 6. The zeros of the numerator and denominator are \( x = -2 \), \( x = 0 \) and \( x = 1 \), so the intervals we have to consider are \( (-\infty, -2) \), \( (-2, 0) \), \( (0, 1) \) and \( (1, \infty) \).

- \( R(-3) = -\frac{3}{4} < 0 \), so \( R(x) < 0 \) in \( (-\infty, -2) \).
- \( R(-1) = \frac{1}{2} > 0 \), so \( R(x) > 0 \) in \( (-2, 0) \).
- \( R(1/2) = -\frac{2}{5} < 0 \), so \( R(x) < 0 \) in \( (0, 1) \).
- \( R(2) = \frac{1}{2} > 0 \), so \( R(x) > 0 \) in \( (1, \infty) \).

Step 7. Comment: The horizontal asymptote is not dashed because it is also an axis.

Figure 2: The graph of \( R(x) = \frac{x}{(x-1)(x+2)} \).